

# On the Jacobian conjecture

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**Abstract** We show that the Jacobian conjecture of the two dimensional case is true.

## 0. Introduction

Let  $T = (P(x, y), Q(x, y)) : \mathbf{C}^2(x, y) \rightarrow \mathbf{C}^2(\xi, \eta)$  be a polynomial mapping with the Jacobian  $J(P, Q) = 1$ . Then we will show that  $T \in \text{Aut}(\mathbf{C}^2)$  where  $\text{Aut}(\mathbf{C}^2)$  is the group of automorphisms of  $\mathbf{C}^2$  (Theorem 3.3). By virtue of S. Kaliman [5], in order to prove that  $T \in \text{Aut}(\mathbf{C}^2)$ , it suffices to prove this when the level curve  $\{P = \alpha\}$  is irreducible for every  $\alpha \in \mathbf{C}$ .

From Lemma 2.3 [1], if the following linear partial differential equation where  $u(x, y)$  is an unknown function

$$\frac{\partial(P, u)}{\partial(x, y)} = g(x, y), \quad (1)$$

has an entire solution for every entire function  $g(x, y)$  of  $\mathbf{C}^2$  with  $P(x, y)$  being fixed as above, the level curve  $\{P = \alpha\}$  is simply connected for every  $\alpha \in \mathbf{C}$ . It is now easy to see that  $T \in \text{Aut}(\mathbf{C}^2)$  (cf. Theorem 3.2 in [1]).

It is to be noted that there is a polynomial  $R(x, y)$  whose every level curve is nonsingular, irreducible and not simply-connected by Bartolo, Cassou-Noguès and Velasco [2]. Such an example is shown in [4, p.256]. Thus, presupposing the existence of a polynomial  $Q(x, y)$  with  $J(P, Q) = 1$  is indispensable in solving the above key problem related to the equation (1) in Theorem 2.4.

In the following sections, it is assumed that polynomials  $P(x, y)$  and  $Q(x, y)$  satisfy the condition  $J(P, Q) = 1$  and the level curve  $\{P = \alpha\}$  is irreducible for every  $\alpha \in \mathbf{C}$ . This assumption will be hereafter called the condition (K) also and expressed in such a manner that  $(P, Q)$  satisfies the condition (K), for example.

## 1. Preliminary

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It is easy to see that the polynomial  $Q(x, y)$  is primitive, namely all of level curves are irreducible except for a finite number of them because  $Q(x, y)$  satisfies  $J(P, Q) = 1$ . It is obvious that  $P(x, y)$  is also primitive. It is obvious also that every level curve of  $Q(x, y)$  is nonsingular.

The following proposition is a summarized statement from Suzuki [7, p.242].

**Proposition 1.1.** *Except for a finite number of exceptional level curves  $\{P = \alpha_i\}_{i=1, \dots, k}$ , every general level curve  $\{P = \alpha\}$  is the same type as  $(g, n)$ . That is, the normalization of  $\{P = \alpha\}$  has a genus  $g$  and  $n$ -punctured boundaries. If we take a sufficiently small disk  $U(\alpha)$  with the center  $\alpha$  which does not contain  $\{\alpha_1, \dots, \alpha_k\}$ , the set  $E_\alpha := \{(x, y); U(\alpha) \ni P(x, y)\}$  is considered to be a topologically trivial fiber space on  $U(\alpha)$  of projection  $P$  with a fiber  $R_0$ , where  $R_0$  is a finite Riemann surface of  $(g, n)$  type.*

Since  $\{P = \alpha_i\}_{i=1, \dots, k}$  is a nonsingular curve, the following proposition, which is well known, is a restatement of Lemma 1 in Nishino-Suzuki [6], for example.

**Proposition 1.2.** *For every relatively compact and connected open set  $K$  on  $\{P = \alpha_i\}$ , there is a relatively compact domain  $M(K)$  in  $\mathbb{C}^2$  such that  $M(K) \subset E_{\alpha_i} := \{(x, y); U(\alpha_i) \ni P(x, y)\}$  where  $U(\alpha_i)$  is a sufficiently small disk with the center  $\alpha_i$  and  $M(K)$  is considered to be a topologically trivial fiber space on  $U(\alpha_i)$  of projection  $P$  with a fiber  $K$ .*

**Proposition 1.3.** *For every  $\alpha \in \mathbb{C}$ , we consider a set  $E_\alpha := \{(x, y); U(\alpha) \ni P(x, y)\}$  as a fiber space such as  $P(x, y) : E_\alpha \rightarrow U(\alpha)$ . If we take a sufficiently small disk  $U(\alpha)$ , there is a global holomorphic section  $L$ , such as  $\{Q(x, y) = \gamma\} \cap E_\alpha$  where  $\{Q(x, y) = \gamma\}$  is irreducible in  $\mathbb{C}^2$  and  $\{Q(x, y) = \gamma\} \cap \{P(x, y) = \delta\} \neq \emptyset$  for every  $\delta \in \mathbb{C}$ .*

**Proof.** Let a point  $(x_0, y_0)$  satisfy  $P(x_0, y_0) = \alpha$  and  $Q(x_0, y_0) = \beta$ . Since  $(P, Q)$  is locally a biholomorphic map of near  $(x_0, y_0)$  to near  $(\alpha, \beta)$ ,  $\{Q(x, y) = \beta'\}$  with  $|\beta - \beta'| < \varepsilon$ , where  $\varepsilon$  is sufficiently small positive number, transversally intersects with  $E_\alpha$  if we take a sufficiently small disk  $U(\alpha)$ . Since  $Q(x, y)$  is primitive as mentioning above, there is a complex number  $\gamma$  such that  $|\beta - \gamma| < \varepsilon$  and  $\{Q(x, y) = \gamma\}$  is irreducible in  $\mathbb{C}^2$ .

It is well known that the image of  $T$  is a Zariski open set. If the complement of the image of  $T$  contains an irreducible algebraic curve such as  $\{C(x, y) = 0\}$ , then  $C(P, Q)$  is a polynomial and  $C(P, Q) \neq 0$ . This contradicts to the fact that  $T$  is a nondegenerate map. Therefore the complement

of the image of  $T$  is a set of finite number of points of  $\mathbf{C}^2(\xi, \eta)$ . And for almost  $\gamma$  with  $|\beta - \gamma| < \varepsilon$ ,  $\{Q(x, y) = \gamma\} \cap \{P(x, y) = \delta\} \neq \emptyset$  for every value  $\delta \in \mathbf{C}$ . Since  $Q(x, y)$  is a primitive polynomial, there is a  $\gamma$  which satisfies a condition of the proposition.  $\square$

Following proposition is easy to see.

**Proposition 1.4.** *Let  $\alpha \in \mathbf{C} - \{\alpha_1, \dots, \alpha_k\}$ . Let  $U(\alpha)$  and  $E_\alpha$  be defined in the same way as Proposition 1.1 and have a global holomorphic section  $L$  in the same way as Proposition 1.3. Then the universal covering space  $\tilde{E}_\alpha$  of  $E_\alpha$  whose base points are points of  $L$  can be thought as  $U(\alpha) \times \tilde{R}_0$  where  $\tilde{R}_0$  is a universal covering surface of  $R_0$ . Roughly speaking,  $\tilde{E}_\alpha$  is a fiber space whose fibers are universal covering surfaces for every fiber of  $E_\alpha$ .*

## 2. Construction of a global solution of the differential equation (1)

We consider a linear partial differential equation

$$\frac{\partial(P, u)}{\partial(x, y)} = g(x, y), \quad (1)$$

where  $u$  is an unknown function and  $g(x, y)$  is an arbitrary entire function of  $\mathbf{C}^2$ . We assume that there is a polynomial  $Q(x, y)$  such that  $(P, Q)$  satisfies the condition (K).

Since (1) is a linear and nonsingular partial differential equation, there is a unique local single-valued holomorphic solution  $u$  near  $L$ , where  $L$  is the same as in Proposition 1.3, having any given holomorphic initial data on  $L$  by Cauchy-Kowalevskaya's theorem (If necessary, we can take  $U(\alpha)$  to be smaller). For applying Cauchy-Kowalevskaya's theorem we remark that  $\{P = \alpha\}$  is the characteristic curve of (1) and a nonsingular curve  $L$  is transversely crossing to  $\{P = \alpha\}$ .

**Lemma 2.1.** *Let  $P(x, y) : E_\alpha \rightarrow U(\alpha)$  be the same as that of Proposition 1.4 and  $u(x, y)$  be the single-valued holomorphic solution of (1) near  $L$  having arbitrarily given holomorphic initial data on  $L$ . Then,  $u(x, y)$  has an analytic continuation along any path in  $E_\alpha$  with any starting point on  $L$ .*

**Proof.** Since the characteristic curve of the equation (1) is  $\{P = \alpha\}$  and the equation (1) can be thought as an analytic family of holomorphic 1-forms such as  $du = \frac{g}{P_y}dx = -\frac{g}{P_x}dy$  where  $P_x dx + P_y dy = 0$  (see p.637 in [1]),  $u(x, y)$

is uniquely fixed as a multi-valued integral of a holomorphic 1-form for every fiber of  $E_\alpha$ .

From Proposition 1.4,  $u(x, y)$  is determined on  $\tilde{E}_\alpha$  as a single-valued function. As  $u(x, y)$  is a holomorphic solution of two variables near  $L$  and holomorphic on every fiber as a function of one complex variable,  $u(x, y)$  is a single-valued holomorphic solution on  $\tilde{E}_\alpha$  by the well-known Hartogs theorem.

Then,  $u(x, y)$  has an analytic continuation along any path in  $\tilde{E}_\alpha$  with the starting point of  $L$ . As any path in  $E_\alpha$  with the starting point of  $L$  can be considered as the one in  $\tilde{E}_\alpha$ , the above lemma holds true.  $\square$

**Lemma 2.2.** *For  $\{P = \alpha_i\}, i \in \{1, \dots, k\}$ , we take a disk  $U(\alpha_i)$  sufficiently small such that  $U(\alpha_i) \cap \{\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_k\} = \emptyset$ ,  $E_{\alpha_i} := \{(x, y); U(\alpha_i) \ni P(x, y)\}$  has a global holomorphic section  $L$  as in Proposition 1.3 and there exists a holomorphic solution  $u(x, y)$  near  $L$  having arbitrarily given holomorphic initial data on  $L$ . Let  $U^*(\alpha_i) = U(\alpha_i) - \{\alpha_i\}$ ,  $L^* = L - \{P = \alpha_i\}$  and  $E_{\alpha_i}^* = \{(x, y); U^*(\alpha_i) \ni P(x, y)\}$ .*

*Then,  $u(x, y)$  can be analytically continued along any path  $l_{rs}$  which starts from a point  $r$  of  $L^*$  and extends to any point  $s$  such that  $l_{rs} \subset E_{\alpha_i}^*$ .*

**Proof.** Let  $\delta_j$  be a disk in  $U^*(\alpha_i)$  such that  $\bigcup_{j=1,2,\dots} \delta_j = U^*(\alpha_i)$  and  $E_{\delta_j} := \{(x, y); \delta_j \ni P(x, y)\}$  is regarded as a topologically trivial fiber space.

Let  $r \in E_{\delta_j} \cap L^*$ . We prove at first for the case  $s \in E_{\delta_k}$ ,  $\delta_j \cap \delta_k \neq \emptyset$  and  $l_{rs} \subset E_{\delta_j} \cup E_{\delta_k}$ .

By the (topological) triviality of  $E_{\delta_j}$  and  $E_{\delta_k}$ , the curve  $l_{rs}$  is homotopic to  $l_{rr'} \cup l_{r's'} \cup l_{s's}$  in  $E_{\delta_j} \cup E_{\delta_k}$  where  $r' \in E_{\delta_j} \cap E_{\delta_k} \cap L^*$ ,  $l_{rr'} \subset L^*$ ,  $l_{r's'} \subset \{P = P(r')\}$  and  $l_{s's} \subset E_{\delta_k}$ .

From Lemma 2.1,  $u(x, y)$ , which has given initial data near  $r' \in L^*$ , can be analytically continued along  $l_{r's'} \cup l_{s's}$  and  $u(x, y)$  has a function element  $u_1$  near  $s$ . As  $l_{rs} \simeq l_{rr'} \cup l_{r's'} \cup l_{s's}$  and  $u(x, y)$  has an analytic continuation along any path in  $E_{\delta_j}$  and  $E_{\delta_k}$  respectively by Lemma 2.1,  $u(x, y)$  can be analytically continued along  $l_{rs}$  and has a function element  $u_1$  near  $s$ .

Next we prove for the case  $s \in E_{\delta_l}$ ,  $\delta_j \cap \delta_k \neq \emptyset$ ,  $\delta_k \cap \delta_l \neq \emptyset$  and  $l_{rs} \subset E_{\delta_j} \cup E_{\delta_k} \cup E_{\delta_l}$ . To simplify the proof (without loss of generality), we assume that  $l_{rs} = l_{rs'} \cup l_{s's''} \cup l_{s''s}$  where  $l_{rs'} \subset E_{\delta_j} \cup E_{\delta_k}$ ,  $l_{s's''} \subset E_{\delta_k} \cup E_{\delta_l}$  and  $l_{s''s} \subset E_{\delta_l}$  by the triviality of  $E_{\delta_j}$ ,  $E_{\delta_k}$  and  $E_{\delta_l}$ .

By the triviality of  $E_{\delta_j}$  and  $E_{\delta_k}$ ,  $l_{rs'} \simeq l_{rr'} \cup l_{r's'}$  where  $r' \in L^* \cap E_{\delta_j} \cap E_{\delta_k}$ ,  $l_{rr'} \subset L^*$  and  $l_{r's'} \subset \{P = P(r') = P(s')\}$ . By the triviality of  $E_{\delta_k}$  and  $E_{\delta_l}$ ,  $l_{rs'} \cup l_{s's''} \simeq l_{rr'} \cup l_{r's'} \cup l_{s's''} \simeq l_{rr'} \cup l_{r'r''} \cup l_{r''s''}$  where  $r'' \in L^* \cap E_{\delta_k} \cap E_{\delta_l}$ ,  $l_{r'r''} \subset L^*$  and  $l_{r''s''} \subset \{P = P(r'') = P(s'')\}$ . Then  $l_{rs} \simeq l_{rr'} \cup l_{r'r''} \cup l_{r''s''} \cup l_{s''s}$  where  $l_{s''s} \subset E_{\delta_l}$ . For the same reason as above,  $u(x, y)$  can be analytically

continued along  $l_{rs}$ . Based on the successive discussions as above, the above lemma is proved easily.  $\square$

**Lemma 2.3.** *Let  $E_{\alpha_i}$  be the same of Lemma 2.2. Then,  $u(x, y)$  can be analytically continued along any path  $l_{rs}$  which starts a point  $r$  of  $L$  to every point  $s$  of  $E_{\alpha_i}$  where  $l_{rs} \subset E_{\alpha_i}$ .*

**Proof.** From the remark made at the outset of this section and with the same reasoning used for the proof of Lemma 2.1, we remark that  $u(x, y)$  can be analytically continued along any path on  $\{P = \alpha_i\}$  as a multi-valued holomorphic function of two variables having given initial data near  $L$ .

Let  $r \in L, s \in E_{\alpha_i}$  and  $l_{rs} = ((x(t), y(t)), t \in [0, 1]$ , where  $r = (x(0), y(0))$  and  $s = (x(1), y(1))$ . Then  $\eta(t) = Q(x(t), y(t))$  is a continuous path on  $\eta$ -plane. We lift up  $\eta(0)$  to  $r_0 \in \{P = \alpha_i\} \cap L$  by  $T^{-1}$  and lift up continuously  $\eta(t)$  to the point of  $\{P = \alpha_i\}$  by  $T^{-1}$ . We set such a path to  $l_{r_0s_0}$ . More precisely speaking,  $\eta = Q(x, y)|_{\{P=\alpha_i\}}$  takes every value at same times except at most finite values  $\gamma_1, \gamma_2, \dots, \gamma_m$ . The map  $T^{-1}$  is able to continue holomorphically along any path from  $r_0$  on  $\{P = \alpha_i\}$  avoiding finite points.

From Proposition 1.2, if we take a relatively compact connected open set  $K$  on  $\{P = \alpha_i\}$  in which  $l_{r_0s_0}$  is contained as a relatively compact subset, we can take a tubular neighborhood  $M(K)$  of  $K$ . Then, the solution  $u(x, y)$  of (1) having given initial data on  $L \cap M(K)$  can be analytically continued along every path in  $M(K)$  if we take  $M(K)$  sufficiently thin with the same reasoning as in Lemma 2.1 using Lemma 1.2.

There are two cases to be considered.

(1) If  $s \in E_{\alpha_i}^*$ , the path  $l_{rs}$  is homotopic to  $l_{rr'} \cup l_{r's'} \cup l_{s's}$  where  $r' \in L^* \cap M(K), l_{rr'} \subset L, l_{r's'} \subset M(K) - \{P = \alpha_i\}$  and  $l_{s's} \subset E_{\alpha_i}^*$  in the same way as one in the proof of Lemma 2.2. From Lemma 2.2,  $u(x, y)$  can be analytically continued along  $l_{rr'} \cup l_{r's'} \cup l_{s's}$  and  $u(x, y)$  has a function element  $u_1$  near  $s$ . Then  $u(x, y)$  can be analytically continued along  $l_{rs}$  and  $u(x, y)$  has a function element  $u_1$  near  $s$ .

(2) If  $s \in \{P = \alpha_i\}$ , the path  $l_{rs}$  is homotopic to  $l_{rr'} \cup l_{r's}$  where  $r' \in L \cap \{P = \alpha_i\}, l_{rr'} \subset L, l_{r's} \subset \{P = \alpha_i\} \cap M(K)$ . From the property of  $M(K)$ ,  $u(x, y)$  can be analytically continued along  $l_{rr'} \cup l_{r's}$  and  $u(x, y)$  has a function element  $u_1$  near  $s$ . Then  $u(x, y)$  can be analytically continued along  $l_{rs}$  and  $u(x, y)$  has a function element  $u_1$  near  $s$ .  $\square$

**Theorem 2.4.** *For every entire function  $g(x, y)$  of  $\mathbf{C}^2$ , the equation (1) has an entire solution  $u(x, y)$ .*

**Proof.** Suppose that  $P(x, y) : E_\alpha \rightarrow U(\alpha)$  satisfies Lemma 2.1 and

suppose that  $P(x, y) : E_{\alpha_i} \rightarrow U(\alpha_i)$  satisfies Lemma 2.3. We take  $\{U(\alpha)\}$  and  $\{U(\alpha_i)\}$  smaller than themselves, if necessary, to satisfy Lemma 2.2 and vary notations of them altogether to be a countable covering  $\{U_j\}_{j=1,2,\dots}$  of  $\mathbf{C}$ . We denote  $E_j$  by  $\{(x, y); U_j \ni P(x, y)\}$ . Let  $L_j$  be a global holomorphic section as in Proposition 1.3 of  $P(x, y) : E_j \rightarrow U_j$  and let  $u_j$  be a solution of Equation (1) having given an initial holomorphic data on  $L_j$  which can be continued analytically along every path in  $E_j$ . Let  $u_j^0$  is a branch of  $u_j$  which coincides to given an initial data on  $L_j$ .

Now we proceed to prove Theorem 2.4 in the following four steps (1) to (4).

(1) At first, consider a case where  $U_j \cap U_k \neq \emptyset$ . We take  $r_j \in L_j \cap E_j \cap E_k$  and  $r_k \in L_k \cap E_k \cap E_j$  and fix a path  $l_{jk}$  from  $r_j$  to  $r_k$  such that  $l_{jk} \subset E_j \cap E_k$  and  $l_{kj} = -l_{jk}$ . We have  $u_j^0$  continue analytically along  $l_{jk}$  from  $r_j$  to  $r_k$  and have it continue analytically on every point of  $L_k \cap E_j$  and we denote such a single-valued branch by  $u_j^k$  since  $L_k \cap E_j$  is simply connected. As  $J(P, u_k^0 - u_j^k) = 0$  in the neighborhood of  $L_k \cap E_j$ ,  $u_k^0 - u_j^k = \varphi_{jk}(P(x, y))$  where  $\varphi_{jk}$  is a single-valued holomorphic function on  $L_k \cap E_j$  and then it is a single-valued holomorphic one on  $U_j \cap U_k$ . Let  $u_k^0$  continue analytically along  $l_{kj}$  from  $r_k$  to  $r_j$  and let it continue analytically on every point of  $L_j \cap E_k$  and we denote such a single-valued branch by  $u_k^j$  since  $L_j \cap E_k$  is simply connected. Similar to the above,  $u_k^0 - u_k^j = \varphi_{kj}(P(x, y))$ , where  $\varphi_{kj}$  can be considered as a single-valued holomorphic function on  $U_k \cap U_j$ . If we have  $u_j^0 - u_k^j$  continue analytically along  $l_{jk}$  from  $r_j$  to  $r_k$  and have it continue analytically on  $L_k \cap E_j$ , then its branch on  $L_k \cap E_j$  is  $u_j^k - u_k^0$ . Hence,  $\varphi_{jk} = -\varphi_{kj}$  on  $U_j \cap U_k$ .

(2) When  $U_j \cap U_k \cap U_l \neq \emptyset$ , we will prove that  $\varphi_{jk} + \varphi_{kl} + \varphi_{lj} = 0$  on  $U_j \cap U_k \cap U_l$  where  $\varphi_{jk}, \varphi_{kl}$  and  $\varphi_{lj}$  are determined in the same way as with the case (1). Since  $\varphi_{jk}(P) = u_k^0 - u_j^k = u_k^j - u_j^0$  on  $L_j \cap E_k$ ,  $\varphi_{kl}(P) = u_l^0 - u_k^l = u_l^j - u_k^j$  on  $L_j \cap E_k \cap E_l$  and  $\varphi_{lj}(P) = u_j^0 - u_l^j$  on  $L_j \cap E_l$ ,  $\varphi_{jk} + \varphi_{kl} + \varphi_{lj} = 0$  on  $U_j \cap U_k \cap U_l$ .

Since  $\{\varphi_{jk}; U_j \cap U_k \neq \emptyset\}$  is a Cousin I data from the fact of (1) and above, and  $\mathbf{C}$  is a Cousin I domain, there exist  $\varphi_j \in \mathcal{O}(U_j)$  and  $\varphi_k \in \mathcal{O}(U_k)$  such that  $\varphi_{jk} = \varphi_j - \varphi_k$  on  $U_j \cap U_k \neq \emptyset$ .

(3) We will prove that  $\{u_j^0 + \varphi_j(P(x, y)) \text{ on } E_j\}_{j=1,2,\dots}$  is a connected multi-valued function  $u(x, y)$ . When  $E_j \cap E_k \neq \emptyset$ , we have  $u_j^0 + \varphi_j(P(x, y))$  continue analytically along  $l_{jk}$  to  $L_k \cap E_j$ . Then, near  $L_k \cap E_j$

$$\begin{aligned} & u_k^0 + \varphi_k(P(x, y)) - (u_j^k + \varphi_j(P(x, y))) \\ &= u_k^0 - u_j^k - \{\varphi_j(P(x, y)) - \varphi_k(P(x, y))\} \\ &= \varphi_{jk}(P(x, y)) - \varphi_{jk}(P(x, y)) = 0. \end{aligned}$$

By successively applying the same reasoning as above, we prove the above

fact.

(4) Lastly, we will prove that such multi-valued function  $u(x, y)$  can be analytically continued along every path  $l_{rs}$  in  $\mathbf{C}^2$  where  $r \in L_j$  and  $s \in E_k$  where  $j$  and  $k$  are arbitrary positive integers. We consider a new  $u(x, y)$  where all function elements of  $\{u_j^0 + \varphi_j(P(x, y)) \text{ on } E_j\}_{j=1,2,\dots}$  are continued analytically to every path in  $\mathbf{C}^2$  if possible.

It is easy to see that  $l_{rs} = l_{rs_1} \cup l_{s_1s_2} \cup \dots \cup l_{s_ns}$  where  $l_{rs_1} \subset E_j \cup E_{k_1}$ ,  $l_{s_1s_2} \subset E_{k_1} \cup E_{k_2}$ ,  $s_1 \in E_{k_1}$ ,  $\dots$ ,  $l_{s_ns} \subset E_{k_n} \cup E_k$ ,  $s_n \in E_{k_n}$ . By the same way of the proof of Lemma 2.3 using  $Q(x, y)$  and  $T^{-1}$ ,  $l_{rs_1}$  is homotopic to  $l_{rr'} \cup l_{r's'} \cup l_{s's''} \cup l_{s''s_1}$ , where  $l_{rr'} \in L_j$ ,  $l_{r's'} \subset \{P(x, y) = P(r')\} \subset E_j \cap E_{k_1}$ ,  $s' \in L_{k_1}$ ,  $l_{s's''} \subset L_{k_1}$  and  $l_{s''s_1} \subset E_{k_1}$ . Since  $u(x, y)$  can be continued analytically along  $l_{rr'} \cup l_{r's'} \cup l_{s's''} \cup l_{s''s_1}$  by Lemma 2.1 and Lemma 2.3. By successively applying the same reasoning as above, we can continue  $u(x, y)$  analytically along  $l_{rs}$ .

Therefore, such  $u(x, y)$  is a connected multi-valued function and can be continued analytically along every path in  $\mathbf{C}^2$ . From the monodromy theorem,  $u(x, y)$  is a single-valued entire function.  $\square$

### 3. Conclusion

Lemma 3.1 (cf. Lemma 2.3 in [1]). *The level curve  $\{P = \alpha\}$  for every  $\alpha \in \mathbf{C}$  is simply-connected.*

Proof. If  $S_0 := \{P = \alpha_0\}$  is not simply-connected, there is a holomorphic 1-form  $a(x)dx$  or  $b(y)dy$ , where  $b(y) = a(x)(-\frac{P_y}{P_x})$  whose integral on  $S_0$  is a multi-valued function by Behnke-Stein theorem in [3]. If we set  $a(x)P_y = -b(y)P_x$ , it represents a holomorphic function  $g_0$  on  $S_0$ . As a consequence of Cartan's Theorem B, there is an entire function  $g$  of  $\mathbf{C}^2$  such that  $g|_{S_0} = g_0$ . It is easy to see that the equation (1) for such  $g$  never has any single-valued entire solution. This contradicts Theorem 2.4.  $\square$

Theorem 3.2 (cf. Theorem 3.2 in [1]). *If the polynomial map  $T = (P(x, y), Q(x, y))$  satisfies the condition (K), then  $T \in \text{Aut}(\mathbf{C}^2)$ .*

Proof. If we restrict  $Q(x, y)$  to  $S := \{P(x, y) = \alpha\}$ ,  $dQ$  has a pole of order 2 at  $\infty$  because the degree of  $dQ = -2$  by the Riemann-Roch theorem and does not vanish on  $S$ . Then  $P|_S$  takes every value once by the residue theorem. It is true for every  $\alpha \in \mathbf{C}$  and therefore  $T \in \text{Aut}(\mathbf{C}^2)$ .  $\square$

By virtue of Kaliman [5], the Jacobian conjecture of the two dimensional case is proved, that is,

Theorem 3.3. *If the polynomial map  $T = (P(x, y), Q(x, y))$  satisfies  $J(P, Q) = 1$ , then  $T \in \text{Aut}(\mathbf{C}^2)$ .*

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